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Characterisation of forests with trivial game domination numbers

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Abstract In the domination game, two players, the Dominator and Staller, take turns adding vertices of a fixed graph to a set, at each turn increasing the number of vertices dominated by the set, until the final set A_* dominates the whole graph. The Dominator plays to minimise the size of the set A_* while the Staller plays to maximise it. A graph is D-trivial if when the Dominator plays first and both players play optimally, the set A_* is a minimum dominating set of the graph. A graph is S-trivial if the same is true when the Staller plays first. We consider the problem of characterising D-trivial and S-trivial graphs. We give complete characterisations of D-trivial forests and of S-trivial forests. We also show that 2-connected D-trivial graphs cannot have large girth, and conjecture that the same holds without the connectivity condition.

Keywords Domination number · Domination game · Game domination number · Tree

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1 Introduction

For a (finite simple) graph G, a subset $A \subset V(G)$ of the vertices of G *dominates* another subset $Z \subset V(G)$ (or the subgraph of G induced by Z) if every vertex $z \in Z$ is either in A, or adjacent to a vertex in A. A is a *dominating set of* G if it dominates G. The *domination number* $\gamma(G)$ of G is the minimum size of a dominating set. There are numerous papers about this well known graph parameter; we restrict our attention to the so called 'Domination Game', introduced recently in Brešar et al. (2010) and since elaborated on in Brešar et al. (2014), Brešar et al. (2015), Brešar et al. (2013), Dorbec et al. (2015), Henning et al. (2014), Kinnersley et al. (2013), and Košmrlj (2014).

The *domination game* is played by two players, a Dominator (D), and a Staller (S), on a graph G. The players take turns choosing vertices of G to add to a set A, and stop when A is a dominating set A_* of G. The one rule is that a player may not add a vertex v to A if it does not increase the size of the set dominated by A, that is, v cannot be added to A if the *closed neighbourhood* of v—the set made up of v and its neighbours—is already dominated. The Dominator's objective is to minimise the size of the final dominating set A_*, while the Staller's objective is to maximise it.

In the *D-first domination game*, D adds the first vertex to A, while in the *S-first domination game*, S adds the first vertex. The *game domination number of* G, $\gamma_g(G)$, is the size of the final dominating set A_* of the D-first domination game played on a graph G when both players play optimally. The size of the final set A_* in an optimal S-first domination game is denoted by $\gamma_g'(G)$.

It was observed in Brešar et al. (2010) that

$$\gamma(G) \le \gamma_g(G) \le 2\gamma(G) - 1,$$

and shown that for every $k \ge 2$ and ℓ with $k \le \ell \le 2k - 1$, there is a graph G such that $\gamma(G) = k$ and $\gamma_g(G) = \ell$.

A graph G is *D-trivial* if $\gamma(G) = \gamma_g(G)$, and *S-trivial* if $\gamma(G) = \gamma_g'(G)$. D-trivial and S-trivial graphs are the object of this work.

It was shown in Kinnersley et al. (2013), verifying a conjecture of Brešar et al. (2010), that $\gamma_g(G)$ and $\gamma_g'(G)$ can differ by at most one for any graph G. One might expect that $\gamma_g(G)$ is always at most $\gamma_g'(G)$, but examples such as the 5-cycle C_5 show that this need not be true: one easily sees that $\gamma(C_5) = 2 = \gamma_g'(C_5)$ but $\gamma_g(C_5) = 3$. There are other examples as well. For any cycle C_n, there are explicit formulas for $\gamma_g(C_n)$ and $\gamma_g'(C_n)$, stated in Košmrlj (2014) and attributed to the unpublished manuscript (Kinnersley et al. 2012). Using these, one gets that $\gamma_g'(C_n) < \gamma_g(C_n)$ exactly when $n \ge 5$ is congruent to 1 or 2 modulo 4. Still more examples are given in Sect. 6.

Notwithstanding examples of graphs G for which $\gamma_g'(G) < \gamma_g(G)$, it was shown in Kinnersley et al. (2013) that when F is a forest, $\gamma_g(F) \le \gamma_g'(F)$ always holds. So every S-trivial forest is also D-trivial.

Our main theorem, Theorem 4.1, gives a characterisation of D-trivial trees. Theorem 4.3, a characterisation of S-trivial trees follows relatively simply from known results. In Fact 2.1, we observe how results from Kinnersley et al. (2013) similar to those stated above will yield characterisations of D-trivial and S-trivial forests from characterisations of D-trivial and S-trivial trees. The characterisations for forests are stated formally in Corollary 4.4.

The layout of the paper is as follows. In Sect. 2 we introduce notation and any known results that we will use. In Sect. 3 we develop some ideas about minimal dominating sets of graphs, which are independent of the Domination game, but that will be useful in our proofs. In Sect. 4 we state and prove our characterisations of S-trivial and D-trivial forests, and give a couple of corollaries. As large girth graphs are somewhat tree-like, the ideas developed in the paper allow us to easily say something about D-trivial graphs of large girth. We expect there should be none. In Sect. 5 we show that there are no 2-connected D-trivial graphs of girth 9 or more, and in Sect. 6 we conjecture that same is true without 2-connectedness. Several other questions are also raised in Sect. 6.

2 Background notation and results

We start with some basic definitions from Brešar et al. (2010) and Kinnersley et al. (2013). Extending the admitted informality in our description of the domination game, we say that a player 'plays' a vertex to mean they add it to the set A. If the closed neighbourhood of a vertex is already dominated by A, we call the vertex 'unplayable'. Otherwise it is 'playable'. Often D will need to play so that a vertex x can never be added to A. This entails dominating its closed neighbourhood without playing x. When he does this, we say he 'blocks' x.

We refer to any minimum dominating set of a graph G as an md-*set of* G, and to any vertex that is in an md-set of G as an md-*vertex*. For a subset V of vertices of G, a subset X of vertices of G is an md-*set of* $G|V$ if it dominates $G - V$ and is a minimum such set in $V(G)$. The size of an md-set of $G|V$ is denoted $\gamma(G|V)$. In the case that $V = \{v\}$, we write $G|v$ for $G|\{v\}$.

Analogous to $\gamma_g(G)$, $\gamma_g(G|V)$ is the size of the final dominating set of an optimally played D-first domination game on a graph G in which the vertices of V are considered dominated from the start of the game. The value $\gamma_g'(G|V)$ is similarly defined for the S-first game.

The graph on which we play the domination game will henceforth be denoted with by $\mathbb{G}$, or $\mathbb{T}$ in the case that it is a tree. The letters G and T, with or without subscripts, will be subgraphs of $\mathbb{G}$ or $\mathbb{T}$, or will be used for generic definitions.

A *D-win strategy* for either a D-trivial or an S-trivial graph $\mathbb{G}$ is a strategy (which one can view as a function f from the power-set of $V(\mathbb{G})$ to $V(\mathbb{G})$ indicating which vertex $f(A)$ of $\mathbb{G}$ the Dominator should add to A) such that when D plays according to the strategy the final dominating set A_* will have size $\gamma(\mathbb{G})$. An *S-win strategy* yields a final dominating set A_* of size greater than $\gamma(\mathbb{G})$. A D-first vertex is the first vertex played by D in a D-win strategy for the D-first game. Generally, we fix a D-win strategy, and denote its D-first vertex by a_0. To show that a given graph is not D-trivial, we will provide an S-win strategy. In doing so, we will informally say that 'S wins', meaning we have given such a strategy.

We finish this section with some useful results that are either trivial or follow trivially from known results. The first is immediate from Kinnersley et al. (2013).

Fact 2.1 *A forest is S-trivial if and only if all its components are S-trivial. A forest is D-trivial if and only if all its components are D-trivial, and all but at most one are S-trivial.*

Proof We prove only the second statement, as the first is even more immediate. The necessity of the condition is clear. Indeed, the component that D plays on first must be D-trivial, and as S can then play first on any other component, all other components must be S-trivial.

The sufficiency of the condition uses the fact, from Kinnersley et al. (2013), that $\gamma_g(T|V) \leq \gamma_g'(T|V)$ for any tree T and any subset V of its vertices. The D-win strategy is to play first on the component that is not S-win, if it exists, (and on any component otherwise), and then afterwards to simply play after S, on the component on which she played, according to his winning strategy on that component. If this is impossible, then D can play on any playable component; and here the fact that $\gamma_g(T|V) \leq \gamma_g'(T|V)$ ensures that he can still win on that component. □

The formula $\gamma_g'(P_n) = \lceil n/2 \rceil$, where P_n is the path on n vertices, is given in Košmrlj (2014) and attributed to Kinnersley et al. (2012). As one can easily verify the formula $\gamma(P_n) = \lceil n/3 \rceil$, this gives the following fact.

Fact 2.2 *The path P_n is S-trivial if and only if $n = 1, 2$ or 4.* □

We cannot find the following simple observation in any of our references, but it likely exists in some form. A set of vertices in a graph, all of which have the same closed neighbourhood, is a *set of clones*. For a graph $\mathbb{G}$ let $\mathrm{cf}(\mathbb{G})$ be the *clone-free reduction* of $\mathbb{G}$, the induced subgraph that we get from removing all but one vertex from every set of clones.

It is easy to see that in the domination game, clones become dominated at the same time, and unplayable at the same time. So adding or removing them has no effect on γ, γ_g or γ_g'. Thus the following is clear.

Fact 2.3 *For any graph $\mathbb{G}$, $\gamma(\mathbb{G}) = \gamma(\mathrm{cf}(\mathbb{G}))$, $\gamma_g(\mathbb{G}) = \gamma_g(\mathrm{cf}(\mathbb{G}))$, and $\gamma_g'(\mathbb{G}) = \gamma_g'(\mathrm{cf}(\mathbb{G}))$.* □

This is not an issue in our main results, as triangle-free graphs, and trees in particular, are clone-free; but it does imply that for every D-trivial or S-trivial graph we find, one gets many more by adding clones. This is used in Corollary 4.6.

3 Minimum dominating sets

It is clear that any vertex in an S-trivial graph $\mathbb{G}$ must be in some minimum dominating set, and that the same is true of any vertex of a D-trivial graph, except possibly for leaves adjacent to the D-first vertex a_0. With this observation that they are important, we collect in this section some simple facts about md-*sets* and md-*vertices*. They greatly streamline our later proofs. These facts are independent of the domination game.

Definition 3.1 For any vertex x in $\mathbb{G}$ let $G_1, \ldots, G_k$ be the components of $\mathbb{G} - x$, and for $i = 1, \ldots, k$, let V_i be the neighbourhood of x in G_i. The pairs (G_i, V_i) are the x-*atoms* of $\mathbb{G}$. If $V_i = \{v_i\}$ for some i, we write (G, v_i) for $(G, \{v_i\})$.

Note that we will often consider the value $\gamma(G|V)$ of such graphs, but we write (G, V) as distinct from $G|V$ to specify that V is the neighbourhood in G of some vertex x not in G. The graph $(G, V) + x$ is the graph we get from G by adding the vertex x that is adjacent to exactly the vertices in V. An x-atom (G, V) is *D-trivial* if in the S-first game on $(G, V) + x$ in which the S-first vertex is x, D has a winning strategy. Similarly, it is *S-trivial* if D has a winning strategy in the D-first game on $(G, V) + x$ with x as the D-first vertex.

Definition 3.2 An x-atom (G, V) of $\mathbb{G}$ is *critical* if

$$\gamma(G|V) < \gamma((G, V) + x),$$

and is *strongly critical* if

$$\gamma(G|V) < \gamma(G).$$

We solidify the definition with an example which we will call on later.

Example 3.3 Let P_n be the path $v_1 \sim \cdots \sim v_n$ for $n \geq 1$, and let (P_n, v_1) be an x-atom of some graph $\mathbb{G}$. (P_n, v_1) is strongly critical if and only if n is congruent to 1 modulo 3. Indeed, it is easy to see that $\gamma(P_n) = \lceil \frac{n}{3} \rceil$ and $\gamma(P_n|v_1) = \lceil \frac{n-1}{3} \rceil$ for any n. Thus $\gamma(P_n|v_1) < \gamma(P_n)$ if and only if n is congruent to 1 modulo 3.

Lemma 3.4 *If x is an* md-*vertex of* $\mathbb{G}$ *then either*

- *all x-atoms (G_i, V_i) are critical, or*
- *some x-atom (G_i, V_i) is strongly critical.*

Proof We prove the contrapositive. Assume that there is a non-critical x-atom (G_1, V_1) and that none of the x-atoms are strongly critical. Then $\gamma((G_1, V_1) + x) = \gamma(G_1|V_1)$ and for all i we have $\gamma(G_i) = \gamma(G_i|V_i)$. Let X_1 be an md-set of $(G_1, V_1) + x$ and for $i > 1$ let X_i be an md-set of G_i. Then $X = \cup X_i$ is an md-set of $\mathbb{G}$ with size

$$\gamma((G_1, V_1) + x) + \sum_{i>1} \gamma(G_i) = \sum_{i\geq 1} \gamma(G_i|V_i)$$

Any dominating set of $\mathbb{G}$ containing x has size at least $1 + \sum_{i\geq 1} \gamma(G_i|V_i)$, so cannot be minimum. $\square$

As any strongly critical x-atom is critical, this gives the following.

Corollary 3.5 *If x is an* md-*vertex of* $\mathbb{G}$, *then* $\mathbb{G}$ *contains a critical x-atom.* $\square$

Definition 3.6 A set of vertices is *incompatible* if it is not contained in any md-set. Otherwise it is *compatible*. We often write that x is incompatible with V or with v if $\{x\} \cup V$ or $\{x\} \cup \{v\}$ are incompatible.

Lemma 3.7 *If (G, V) is a critical x-atom of $\mathbb{G}$, then x is incompatible with any set Y of vertices of G which dominates V and contains a vertex of V. In particular if $V = \{v\}$ then x and v are incompatible.*

Proof Towards contradiction, assume there is an md-set X of $\mathbb{G}$ containing $\{x\} \cup Y$ for some dominating set $Y \subset V(G)$ of V which contains $v \in V$. Observe that $X' = X \cap V(G)$ dominates $(G, V) + x$. Indeed, it clearly dominates $G - V$; as it contains Y it also dominates V, and so dominates G; and as it contains $v \in V$, it dominates x. Thus, as (G, V) is a critical x-atom, X' has size at least $\gamma(G|V) + 1$.

Now let Z be an md-set of $G|V$. It has size $\gamma(G|V)$, so the set $(X - X') \cup Z$ is one vertex smaller than X. Further, $(X - X') \cup Z$ is a dominating set of $\mathbb{G}$. Indeed, $(X - X')$ clearly dominates every vertex not in $(G, V) + x$, Z dominates $G - V$, and $x \in (X - X')$ dominates $V \cup \{x\}$.

Thus $(X - X') \cup Z$ is a dominating set of $\mathbb{G}$ that is smaller than X. This is a contradiction. □

Essentially the same proof gives the following.

Lemma 3.8 *Let (G, V) be a strongly critical x-atom of $\mathbb{G}$, and U be the set of vertices in $G - V$ dominated by V. Then x is incompatible with any subset $U' \subset U$ which dominates V. In particular, if $V = \{v\}$, then x is incompatible with any neighbour u of v.*

Proof Towards contradiction, assume there is an md-set X of $\mathbb{G}$ containing $\{x\} \cup U'$ for some $U' \subset U$ which dominates V. Then $X' = X \cap V(G)$ dominates G, so by the strong criticality of (G, V), it has size at least $\gamma(G|V) + 1$. But then where Z is an md-set of $G|V$, we have, as in the proof of Lemma 3.7, that $(X - X') \cup Z$ is a smaller dominating set of $\mathbb{G}$ than X. This is a contradiction. □

Lemma 3.9 *For any non-isolated* md-*vertex x in a graph $\mathbb{G}$, there is an incompatible neighbour x'.*

Proof By Corollary 3.5 there is a critical x-atom (G, x'), and by Lemma 3.7 x and x' are incompatible. □

Definition 3.10 Whether or not $\mathbb{G}$ is a tree, we call a vertex of degree 1, a *leaf*.

Corollary 3.11 *Let x be an* md-*vertex adjacent to a vertex x' which is adjacent to a leaf ℓ different than x. There is a strongly critical x-atom (G, v) not containing x'.*

Proof As x' is adjacent to a leaf ℓ, any md-set can be assumed to contain it (by replacing ℓ in the md-set with x' if x' is not in the set). So as x is an md-vertex, x and x' are compatible. Thus by Lemma 3.7, the x-atom containing x' is not critical. But then by Lemma 3.4, some other x-atom is strongly critical. □

4 Characterisation of *D*-trivial and *S*-trivial forests

We start this section with our main theorem.

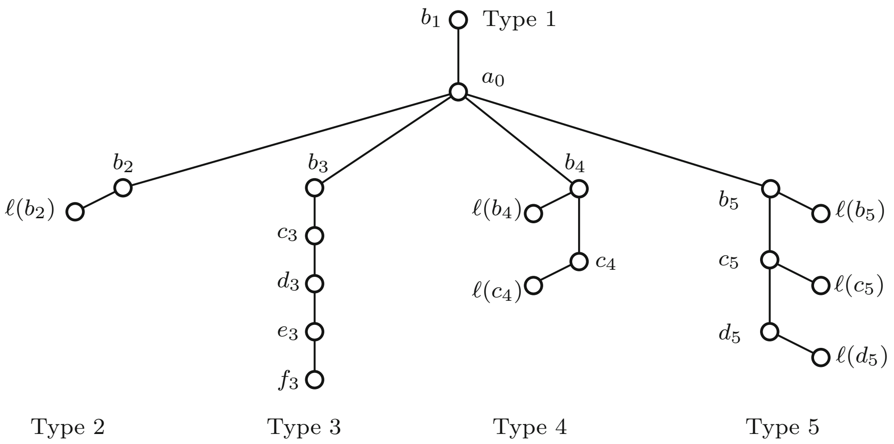

Fig. 1 A D-trivial tree with one a_0-atom of each type

Theorem 4.1 *Let $\mathbb{T}$ be D-trivial tree. Then $\mathbb{T}$ consists of a vertex a_0 and a set of d a_0-atoms, for some integer $d \geq 0$, such that any a_0-atom (B, b) is one of the following types.*

Type 1: *A single vertex.*
Type 2: *A 2-path. (i.e., a path of 2 vertices.)*
Type 3: *A 5-path in which b is a leaf.*
Type 4: *A 3-path in which b is adjacent to a leaf.*
Type 5: *A tree constructed from a 3-path with a leaf b, by adding a new leaf to each vertex.*

Further, unless $\mathbb{T}$ is just a_0, at least one of the a_0-atoms is of Type 1.

See Fig. 1 for the D-trivial tree with five a_0-atoms, one of each type. The figure assigns labels to the vertices of these a_0-atoms which will be useful in the proof of the theorem. Our proof of Theorem 4.1 uses the following technical lemma which allows us to avoid the repetition of tedious arguments.

Lemma 4.2 *Let $\mathbb{G}$ be a D-trivial graph, and let a_0 be a D-first vertex. Then the following are true.*

i. *No vertex x except possibly a_0 is adjacent to more than one leaf.*
ii. *Every vertex $x \neq a_0$ has $\deg(x) \leq 3$, and if $\operatorname{dist}(x, a_0) \geq 3$ then $\deg(x) \leq 2$.*
iii. *It (T, v_1) is a strongly critical x-atom not containing a_0, then either (T, v_1) is a single vertex; or T is a P_4 with endpoint v_1. Further, in the case that T is a P_4, the following are true.*
   (a) *The vertex x has degree 2, and*
   (b) *if x is not adjacent to a_0, then its second neighbour (different from v_1) does not have a leaf.*

First we use Lemma 4.2 to prove Theorem 4.1. Following this, we prove Lemma 4.2.

Proof (of Theorem 4.1) Let $\mathbb{T}$ be a D-trivial tree and a_0 be a D-first vertex. If $\mathbb{T}$ has no other vertices, then we are done, so we may assume that a_0 has a neighbour ℓ. By Lemma 3.9, ℓ is incompatible with x. So that S cannot play ℓ on her first turn, it must be a leaf. This gives us the 'Furthermore' part of the theorem.

Now let L be the set of leaves adjacent to a_0 and let (B, b) be an a_0-atom in $\mathbb{T} - L$. Our task now is to show that (B, b) is of one the Types (2)–(5) listed in Theorem 4.1.

By Corollary 3.11 there is a strongly critical b-atom (T, v_1) not containing a_0. If b has degree 2, then by Lemma 4.2(iii.) either (T, v_1) is a single vertex, and so (B, b) is of Type 2, or (T, v_1) is a P_4 with leaf v_1, and so (B, b) is of Type 3. Either way, we are done. Thus we may assume that b has degree 3 and so by Lemma 4.2(iii.a.) (T, v_1) is a single vertex, so a leaf of $\mathbb{T}$. To stay consistent with Fig. 1, relabel v_1 as $\ell(b)$, and let c be the third neighbour of b.

As $c \sim b \sim \ell(b)$, we have by Corollary 3.11 that there is a strongly critical c-atom (T', v_1') not containing b, so not containing a_0. If c has degree 2 then by Lemma 4.2(iii.) (T', v_1') is a single vertex, and so (B, b) is of Type 4, and we are done, or (T', v_1') is a P_4. But as b has the leaf $\ell(b)$ this latter cannot happen by Lemma 4.2(iii.b.). So we may assume that c has degree 3. Using Lemma 4.2(iii.), we assert that c has a leaf $\ell(c)$ and another neighbour d.

By Lemma 4.2(i), d has another neighbour $\ell(d)$, and by 4.2(ii) it has degree 2. As above, Lemma 4.2(iii.b.) assures us that $\ell(d)$ is a leaf. So (B, b) is of Type 5.

We have shown that $\mathbb{T}$ is of the form given in the statement of the theorem. To finish off, we must observe that all such trees are indeed D-trivial. This follows by showing for each a_0-atom (B, b), that $\gamma_g'(B|b) = \gamma(B|b)$. Indeed, if this is true, then playing a_0 first, D's strategy is then to follow S, playing on the same a_0-atom she does, if possible, with his winning strategy on that a_0-atom. Again, by the fact from Kinnersley et al. (2013) that $\gamma_g(T|V) \leq \gamma_g'(T|V)$ for any tree T and subset V of its vertices, we have that D still wins if he must play first on some a_0-atom.

For $i = 1, \ldots, 5$, let (B_i, b_i) be an a_0-atom of Type i, with vertices labelled as in Fig. 1. It will be enough to show that for each $i = 1, \ldots 5$,

$$\gamma_g'(B_i|b_i) = \gamma(B_i|b_i).$$

In the case $i = 1$ this is trivially true because b_1 is unplayable. In the case $i = 2$, $\ell(b_2)$ must be dominated, so $\gamma(B_2|b_2) = 1$, and when either of the vertices in B_2 are played, the other becomes unplayable. In the case $i = 3$, $\gamma(B_3|b_3) = 2$, and whatever S plays, D can dominate with some second vertex. In the case $i = 4$, both the leaves $\ell(b_4)$ and $\ell(c_4)$ must be dominated, whichever of the four vertices S plays, D can dominate B_4 with some second vertex. In the case $i = 5$, any dominating set must contain at least three vertices to dominate the leaves. If S plays a non-leaf it is trivial that D can play some second vertex and force a three element dominating set. If S plays $\ell(c_5)$ then D plays d_5, and only $\ell(b_5)$ or b_5 are playable. If S plays any other leaf, then D plays c_5. □

Now we prove the lemma.

Proof (of Lemma 4.2)

Part (i.) is trivial as if a vertex has more than one leave, then neither of the leaves can be an md-vertex. If some vertex other than a_0 has two leaves, then S can play one of them, and win.

For part (ii.), let $x \neq a_0$ have degree $k \geq 2$. By Lemma 3.9, x has an incompatible neighbour x' (which therefore is not a_0). Let S play x' on her first turn. Then D must block her from playing x on her next turn. So for any undominated neighbour u of x, D must dominate u by playing it or a neighbour (different from x). The neighbour x' of x is dominated, and if $\mathrm{dist}(x, a_0) = 2$ then the common neighbour of x and a_0 is dominated, but no other neighbours are. As $\mathbb{T}$ is a tree, D can dominate at most one more, so $k \leq 3$. If $\mathrm{dist}(x, a_0) \geq 3$, then the neighbour of x on the path towards a_0 is not dominated, so only the neighbour x' is. Thus if $\mathrm{dist}(x, a_0) \geq 3$, then $k \leq 2$, as needed.

For part (iii.) let (T, v_1) be a strongly critical x-atom not containing a_0. We may assume that $x \neq a_0$, as in the proof of Theorem 4.1 we already showed that any critical a_0-atom is a leaf. So S can play x on her first move. As (T, v_1) is a strongly critical x-atom in a D-win graph $\mathbb{T}$, this implies that (T, v_1) is D-trivial.

Now, assume that (T, v_1) is not a single vertex. Then v_1 has some neighbour v_2. As (P_2, v_1) is not strongly critical by Example 3.3, v_2 must have some other neighbour v_3. Now both v_1 and v_2 are incompatible with x by Lemmas 3.7 and 3.8, so if S plays x, D must block both v_1 and v_2 with his next move. Thus D must play v_3, and v_1 and v_2 can have no neighbours that have not already been mentioned. By part (ii.) of the lemma, T is then a path $v_1 \sim \cdots \sim v_n$ with $n \geq 2$. By Example 3.3 we have that n is congruent to 1 modulo 3. Moreover, we argued above that if S plays x on her first move, then D plays v_3 on his first move. So the remaining v_3-atom (T', v_4), where T' is the path $v_4 \sim v_5 \sim \cdots \sim v_n$, must be S-trivial. This is clearly only true if v_4 is unplayable, so if $n = 4$. Thus $T = P_4$, as needed.

Now assuming that $T = P_4$ we must show the 'Further' part of part (iii.) of the lemma. As there is an x-atom not containing a_0, x clearly has degree at least 2. To see that it has degree exactly 2, assume, towards contradiction, that it has a third neighbour x' in a third x-atom containing neither v_1 or a_0. We show there is an S-win strategy. Indeed, let S play v_4 on her first move. D must block v_3 on his second move, so he must play v_1 or v_2. As x' is not yet dominated S can play x. But by Lemmas 3.7 and 3.8, x is incompatible with both v_1 and v_2, so S has won. This contradicts the fact that $\mathbb{G}$ is D-trivial, so gives us (iii.a.).

For (iii.b), assume that x has a neighbour $x' \neq a_0$ with a leaf. Let S play x' on her first turn. To block x', D must play x on his second turn. But then S can play v_1 which is incompatible with x. So S wins, which is a contradiction. □

This completes the proof of the lemma, and so of Theorem 4.1. From the proof of the lemma, and Fact 2.2 we immediately get the follows.

Theorem 4.3 *A tree $\mathbb{T}$ is S-trivial if and only if it is an n-path P_n for $n = 1, 2,$ or 4.*

Proof By Fact 2.2 it is enough to show that the only S-trivial trees are paths. But the proof of Lemma 4.2 does this. Indeed, any vertex x in an S-trivial tree $\mathbb{T}$ must be an md-vertex. Such x can have degree at most 2 by the proof of Lemma 4.2(ii) for a vertex having distance at least 3 from a_0. □

Now from Theorems 4.1 and 4.3 we get the following restatement of Fact 2.1.

Corollary 4.4 *A forest is S-trivial if and only if every component is a path P_n for $n = 1, 2,$ or 4. A forest is D-trivial if and only if it is S-trivial or the union of an S-trivial forest and a tree $\mathbb{T}$ from Theorem 4.1.* □

It is easy to check that an a_0-atom (B_i, b_i) of Type i from Theorem 4.1 satisfies $\gamma(B_i|b_i) = |V(B_i)|/2$ if $i = 2, 4,$ or 5, and $\gamma(B_i|b_i) = (|V(B_i)| - 1)/2$ if $i = 1$ or 3. Thus the following is clear.

Corollary 4.5 *For any D-trivial tree $\mathbb{T}$, we have*

$$\gamma_g(\mathbb{T}) = \gamma(\mathbb{T}) = \frac{1 + |V(\mathbb{T})| - p}{2}$$

where p is the number of a_0-atoms of $\mathbb{T}$ of Type 1 or 3. □

In Košmrlj (2014), one can find results stating that there are graphs of arbitrary connectivity and order having given differences in their domination number and game domination numbers. The following analogous statements now follow trivially by Fact 2.3.

Corollary 4.6 *For any $\gamma \geq 1$, $k \geq 1$ and large enough n, there is a graph $\mathbb{G}$ on n vertices with connectivity k such that $\gamma(\mathbb{G}) = \gamma_g(\mathbb{G}) = \gamma$.*

Proof For $k = 1$ this is an immediate corollary of Theorem 4.1, by choosing the appropriate number of branches. For larger k it follows from Fact 2.3 by replacing each vertex of $\mathbb{G}$ in the $k = 1$ case by k clones. The resulting graph is known as the lexicographic product of a tree and a k-clique, and is easily seen to be k-connected. □

5 Graphs without short cycles

Lemma 5.1 *If (G, V) is an S-trivial x-atom of a graph $\mathbb{G}$ with $|V| \geq 2$, then $(G, V) + x$ contains a cycle of length at most 8.*

Proof Recall that (G, V) being S-trivial means that with D-first vertex x, there is a D-win strategy on $(G, V) + x$. Observe also that as (G, V) is an x-atom, so is connected, and as $|V| \geq 2$, $(G, V) + x$ contains a cycle. Our proof is by contradiction; we assume that $(G, V) + x$ has girth at least 9 and show that when D plays x, there is an S-win strategy. Observe that as $V \cup \{x\}$ is incompatible, playing all of V is a winning strategy for S.

If every vertex in V has degree at least 2 (in G, not in $(G, V) + x$), then S has a clear strategy for playing all of V. On each turn, if D's previous play was a vertex in G having distance 1 or 2 (in G) from a vertex $v \in V$, S plays v; otherwise she plays an arbitrary vertex of V. That the girth $(G, V) + x$ is at least 9 assures that D cannot have played anything having distance 1 or 2 to more than one vertex of V.

We may therefore assume that there is some vertex $a \in V$ that has degree 1 in G. Let b be the neighbour of a. As (G, V) is an x-atom, G is connected, so there is a path from b back to V that does not use the edge (b, a). By the girth condition this path has length at least 8 so it begins with a path $b \sim c \sim d$ such that c and d have distance at least 4 from $V - \{a\}$ in G. Now let S play a on her first move. As b is incompatible with $\{x, a\}$ (because x and b are the only neighbours of a), D must block S from playing b on her next move; so D must play c or d. As both of these have distance at least 4 from any other $v_i \in V$, this does not help block any of them, or their neighbours. S continues playing vertices of V having degree 1 in G, until they are exhausted, and then continues on as she played in the case where every vertex of V has degree at least 2. □

This immediately gives the following.

Theorem 5.2 *Any D-trivial graph without leaves has girth at most* 8. *In particular, any* 2*-connected D-trivial graph has girth at most* 8.

Proof Let $\mathbb{G}$ be a D-trivial graph without leaves. By Corollary 3.5, for any D-first vertex a_0 in $\mathbb{G}$, there is a critical a_0-atom (G, V). Clearly it must be S-trivial. If $V = \{v\}$ this means that v is a leaf, as by Lemma 3.7 a_0 and v are incompatible. So as $\mathbb{G}$ has no leaves, we have that $|V| \geq 2$. But then by Lemma 5.1 $(G, v) + a_0$ contains a cycle of length at most 8. □

6 Concluding remarks

With respect to the Domination Game, we have asked a very natural question: when is the game domination number of a graph equal to the domination number of a graph?

From our characterisation of D-trivial and S-trivial trees, we see that D-trivial and S-trivial graphs are quite special. However, the class of them is not finite. Even under the assumptions of being clone-free and connected, there are infinitely many S-trivial graphs. Indeed, any graph G that is vertex transitive and has a dominating set of size two, is S-trivial with $\gamma(G) = 2$; and any graph that is vertex transitive and has a set of size two dominating all vertices but one, is S-trivial with $\gamma(G) = 3$. Examples in the first case are the Cayley graphs on $\mathbb{Z}_{4n+2}$ generated by the set $\{\pm 1, \ldots, \pm n\}$. Clearly for any i the set $\{i, i + 2n + 1\}$ is a dominating set. In the second case there are the Cayley graphs on $\mathbb{Z}_{4n+3}$ generated by the set $\{\pm 1, \ldots, \pm n\}$. The set $\{i, i + 2n + 1\}$ dominates everything but $i - n - 1$. In none of the examples above is the graph also D-trivial.

While a connected D-trivial graph can have an arbitrarily large domination number, as is shown in Theorem 4.1, we cannot find connected S-trivial graphs with domination number larger than 3. Moreover, the graphs from Theorem 4.1 are broken into small components when the D-first vertex a_0 is removed.

This leads us to ask if there can be any connected graphs G that are

(i) connected and S-trivial with $\gamma(G) > 3$, or
(ii) 2-connected, clone-free, and D-trivial with $\gamma(G) > 3$.

We suspect that such graphs cannot exist.

We feel that one should be able to improve the girth 8 in Theorem 5.2 to 7, but as C_7 is D-trivial, one cannot improve it beyond that. Further, our proof shows that the D-first vertex a_0 of a D-trivial graph of girth at least 9 must be a cut vertex. This suggests to us that the 2-connectedness is not needed. We conjecture the following.

Conjecture 6.1 *Any connected D-trivial graph is either a tree or has girth at most* 7.

Proving this seems to require showing that any S-trivial a_0-atom is either a tree or contains a cycle of length less than 8. We wonder if the proof of Lemma 5.1 can be tightened to give that the only a_0-atom of girth 7 is C_7. Whether or not this is true, a characterisation of S-trivial a_0-atoms of girth 7 or 8, so of D-trivial graphs of girth 7 or 8 would be interesting. As would a characterisation of D-trivial and S-trivial graphs of any girth $g \leq 7$.

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